Nuclear Physics B268 (1986) 489-508

THE FISSIONING UNIVERSE:
Topological inflation and Kaluza-Klein cosmologies*

Michio KAKU and Joseph LYKKEN

Physics Dept., City College of the City University of New York, NY 10031, USA

Received 19 June 1984
(Revised 8 October 1985)

We propose a Kaluza-Klein cosmology by reversing the usual scenario: instead of starting with a flat $4+N$ dimensional universe in which N of the dimensions curl up into a compact manifold, we start with a compact $3+N$ dimensional manifold in which 3 of the dimensions are allowed to peel off and expand into the known universe.

We reverse the usual "spontaneous compactification" scenario and begin with a closed manifold M^{3+N} which undergoes "spontaneous fissioning" into a product manifold $M^3 \times M^N$. Remarkably, the 3-dimensional universe M^3 can undergo a rapid de Sitter expansion large enough to solve the horizon and flatness problem. We call this "topological inflation", which we propose as an alternative to the usual GUT inflation. The inflationary phase automatically terminates into a big bang phase.

1. Introduction

The Kaluza-Klein theory [1] is one of the most promising and attractive candidates for a unified field theory which includes both the gravitational and particle interactions. By embedding Einstein's theory into a higher $4+N$ dimensional manifold, the Yang-Mills interactions emerge naturally when we decompose the components of the higher dimensional metric tensor. Although the Klein-Kaluza picture is appealing from the gauge theory point of view, the application of Kaluza-Klein theory to cosmology remains totally obscure.

In the usual Kaluza-Klein scenario, we assume that the universe was originally defined on a flat $4+N$ dimensional manifold, in which N of the dimensions underwent spontaneous compactification [2], i.e. N dimensions suddenly curled up into a compact manifold the size of the Planck length, leaving a flat four-dimensional riemannian universe.

In this paper, we apply Kaluza-Klain theory to cosmology by reversing this scenario. Instead of N dimensions curling up into a compact manifold, we assume that the original universe was defined on a $3+N$ dimensional compact manifold M^{3+N} on the scale of the Planck length which suddenly "fissioned" into a product manifold $M^3 \times M^N$. The M^3 universe then exploded outward into the known universe, while the M^N universe collapsed.

* Partially supported by NSF PHY-82-15364 and CUNY-FRAP-RF 13873.

The usual Klein–Kaluza picture assumes spontaneous compactification, where a non-compact manifold suddenly becomes balled up into a compact manifold. We, however, assume the opposite—"spontaneous fissioning [3], i.e. the original compact manifold "fissioned" into two smaller compact manifolds, one of which then exploded rapidly into the known universe while the other compact manifold suddenly collapsed down to the Planck length.

Remarkably, we find that the rapid expansion of the 3-dimensional universe is in an exponential de Sitter phase. With a suitable choice of parameters, we can achieve sufficient exponential expansion ("inflation" [4]) in order to solve the long-standing horizon and flatness problems.

We propose this "topological inflation" as a solution to the horizon and flatness problems. Because the usual GUT inflation picture has certain problems with fitting or fine tuning the parameters [5], this "topological inflation" formalism is an attractive alternative to the usual formalism.

We find that the fissioning takes place in three stages:

(I) In the first stage, the original compact manifold M^{3+N} undergoes a de Sitter expansion in $3+N$ dimensional space. This manifold then suddenly fissions into a product manifold of $M^3 \times M^N$ initiated, for example, by the Freund–Rubin ansatz [6].

(II) In the second stage, the three dimensional manifold undergoes a rapid exponential expansion, while the N-dimensional manifold collapses down to the Planck length. This period of topological inflation can be made sufficiently large to solve the horizon and flatness problems.

(III) In the last phase, the de Sitter expansion automatically shuts itself off and smoothly continues over into a big bang phase (see fig. 1).

The attractive features of our model are as follows:

(i) We have checked explicitly on computer that the M^3 universe undergoes an inflation phase which eventually cuts itself off and goes over into a big bang phase. A suitable choice of parameters can create a sufficient amount of inflation to generate the desired e^{65} factor.

(ii) Although we have to fix the value of the primordial cosmological constant before spontaneous fissioning takes place, we find that our formalism predicts an extremely small but non-vanishing value of the effective cosmological constant when viewed from the present universe. This may help to shed some light on the cosmological constant problem.

(iii) Supersymmetry can be probably included in this model if we use the O(8) model to generate spontaneous fissioning. If we calculate quantum corrections to an O(8) ansatz which preserves supersymmetry, however, first-loop corrections vanish between bosonic and fermionic contributions [7]. Therefore, we must calculate the quantum corrections around a supersymmetry-breaking ansatz. Also, because our formalism generates a series of terms which contribute to the cosmological constant, this may shed some light on the problem of how to convert the supersymmetric anti-de Sitter phase into a de Sitter one [8].

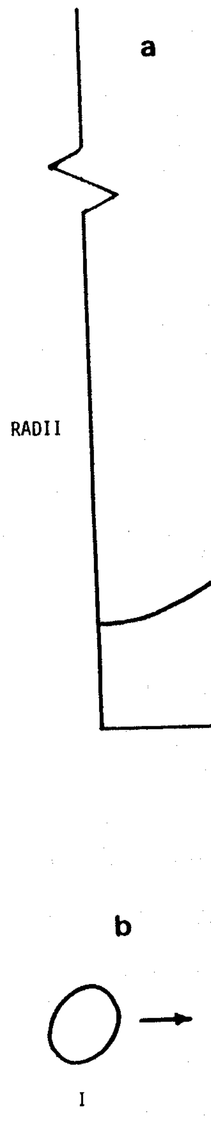

Fig. 1. (a) In region
has fissioned into a
manifold undergoes
III, the 3-dimension
manifold continues
a closed $3+N$ dim
unknown dynamica
inflationary (de Sitt
are driving the infla

ctification, where a
pact manifold. We,
he original compact
ne of which then
manifold suddenly

nsional universe is
arameters, we can
order to solve the

orizon and flatness
oblems with fitting
" formalism is an

dergoes a de Sitter
enly fissions into a
und–Rubin ansatz

undergoes a rapid
apses down to the
sufficiently large to

shuts itself off and

erse undergoes an
a big bang phase.
flation to generate

nological constant
nalism predicts an
ical constant when
on the cosmologi-

f we use the O(8)
um corrections to
t-loop corrections
re, we must calcu-
satz. Also, because
the cosmological
vert the supersym-

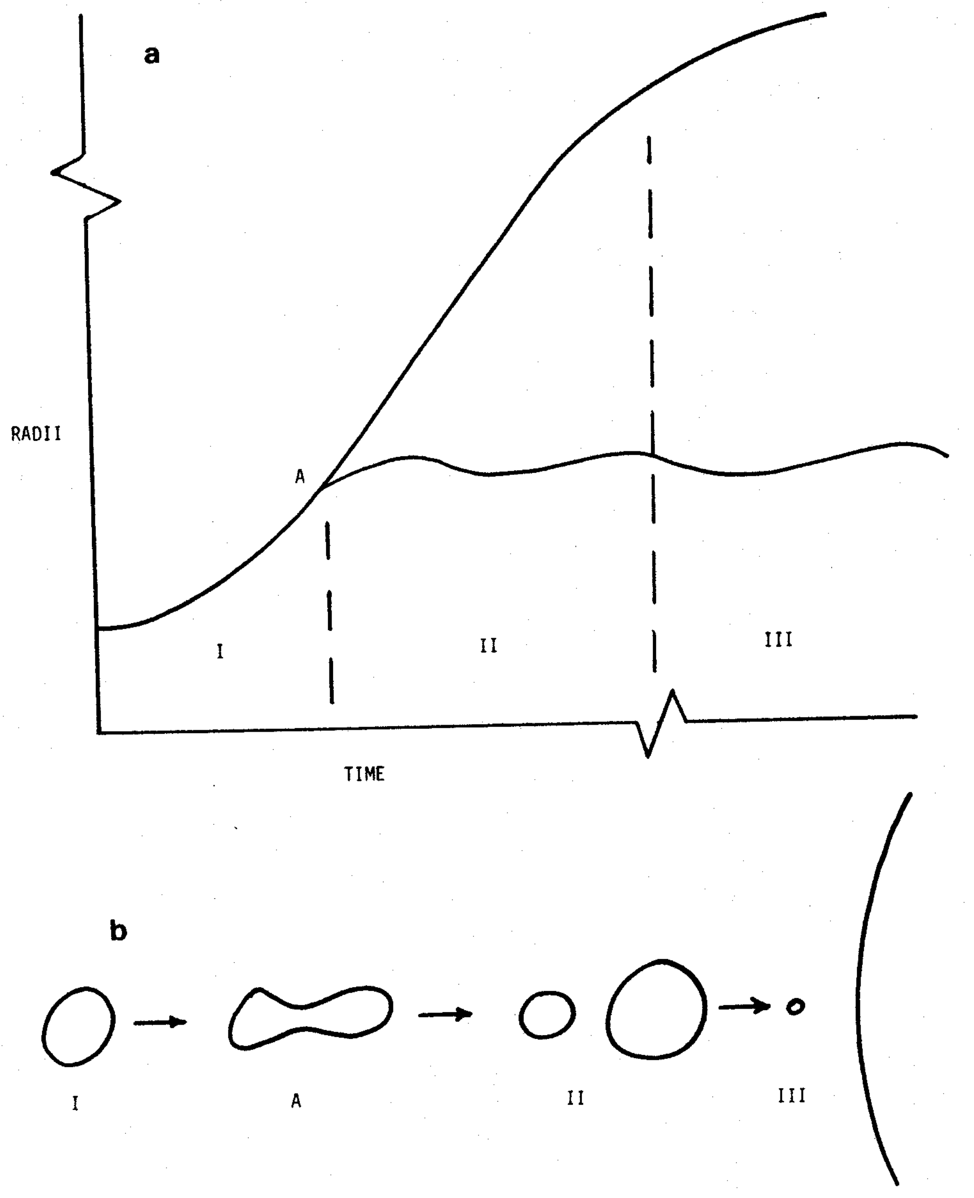

Fig. 1. (a) In region I, the universe is a closed manifold in $3+N$ dimensions. At point A, the universe has fissioned into a closed 3-dimensional manifold and a N-dimensional manifold. The 3-dimensional manifold undergoes a rapid de Sitter expansion, while the N-dimensional manifold collapses. In region III, the 3-dimensional manifold makes a smooth transition to a big bang phase, while the N-dimensional manifold continues to oscillate around its equilibrium configuration. (b) In region I, the universe is still a closed $3+N$ dimensional manifold. At point A, the universe begins to fission due to some as yet unknown dynamical symmetry breaking. In region II, the two manifolds diverge, one undergoing a rapid inflationary (de Sitter) phase, while the other collapses. In region III, the quantum corrections which are driving the inflation forward gradually subside, allowing the universe to smoothly make the transition to a big bang phase.

(iv) Although we performed our calculation on closed spheres S^N, we could also have used other homogeneous manifolds such as CP^N. Using complex manifolds, we may be able to incorporate GUT's into our formalism, although it is independent of GUT's.

(v) Although we use a zeta functional regularization to define our theory, we find that our equilibrium results agree remarkably well with the results of Candelas and Weinberg [9].

We should make perfectly clear in our paper that the precise mechanism by which we break the initial symmetry of the universe, whether by Freund–Rubin type mechanisms or otherwise, is left completely unsolved. This is the fundamental weakness of this paper. As a result, it is difficult to argue precisely how much inflation actually takes place in this model, because in principle any initial values of the radii are possible after fissioning. We have, in a sense, simply pushed back our understanding of the fine tuning problem from the parameters found in the GUT model to the parameters of the original dynamical breaking of the Kaluza-Klein universe.

As a result, we should be careful to state that we certainly have not solved the fine tuning problem. What we are claiming, however, is that we are postulating a new mechanism which (in principle) may solve the fine tuning problem but which (in practice) depends crucially on how the original symmetry of the Kaluza–Klein universe was broken, which nobody knows. We simply wish to correct the conventional wisdom, however, that inflation takes place solely via GUT mechanism or even super-GUT mechanisms. However, we also wish to state that it must be physically possible to experimentally justify the claim that our Universe once went through a Kaluza–Klein universe, and that one of these physically relevant phenomena might be Kaluza-Klain-induced inflation.

2. The fissioning universe

Our starting point is the usual Einstein theory defined on a closed $3+N$ dimensional sphere with a cosmological constant:

$$R_{AB}-\tfrac{1}{2}g_{AB}(R+\Lambda)=-8\pi\bar{G}T_{AB}\,,\qquad A,B=1,2,\ldots,4+N\,. \tag{2.1}$$

Let us assume that we can spontaneously break the original manifold of the theory down to $S^3\times S^N$. We assume that we can now split the metric tensor of the theory into the following:

$$\begin{aligned} &g_{AB}=(-1,\,g_{ij},\,g_{\alpha\beta})\,,\qquad g_{i\alpha}=0\,,\\ &i,j=1,2,3\,,\qquad \alpha,\beta=1,2,\ldots,N\,. \end{aligned} \tag{2.2}$$

Now let us assume a Robertson–Walker ansatz defined over this product manifold. Let us assume that the 3 (N)-dimensional sphere has a radius given by $\rho(\bar{\rho})$. We

easily find that:

$$
\begin{aligned}
&g_{ij} = \rho^2(t)\tilde{g}_{ij}\,, \qquad g_{\alpha\beta} = \bar{\rho}^2(t)\tilde{g}_{\alpha\beta}\,,\\
&R_{00} = 3\ddot{\rho}/\rho + N\ddot{\bar{\rho}}/\bar{\rho}\,,\\
&R_{ij} = \tilde{R}_{ij} - g_{ij}(\ddot{\rho}/\rho + 2(\dot{\rho}/\rho)^2 + N(\dot{\rho}/\rho)(\dot{\bar{\rho}}/\bar{\rho}))\,,\\
&R_{\alpha\beta} = \tilde{R}_{\alpha\beta} - g_{\alpha\beta}(\ddot{\bar{\rho}}/\bar{\rho} + (N-1)(\dot{\bar{\rho}}/\bar{\rho})^2 + 3(\dot{\rho}/\rho)(\dot{\bar{\rho}}/\bar{\rho}))\,,\\
&R = \tilde{R} - 6\bar{\rho}/\rho - 2N\ddot{\bar{\rho}}/\bar{\rho} - 6(\dot{\rho}/\rho)^2\\
&\qquad - N(N-1)(\dot{\bar{\rho}}/\bar{\rho})^2 - 6N(\dot{\bar{\rho}}/\bar{\rho})(\dot{\rho}/\rho)\,.
\end{aligned}
\tag{2.3}
$$

Inserting these identities into Einstein's equations, we find:

$$
\begin{aligned}
&3(\dot{\rho}/\rho)^2 + \tfrac{1}{2}N(N-1)(\dot{\bar{\rho}}/\bar{\rho})^2 + 3N(\dot{\rho}/\rho)(\dot{\bar{\rho}}/\bar{\rho}) + 3/\rho^2\\
&\qquad + \tfrac{1}{2}N(N-1)/\bar{\rho}^2 - \tfrac{1}{2}\Lambda = -8\pi\bar{G}T_{00}g^{00}\,,\\
&2\ddot{\rho}/\rho + (\dot{\rho}/\rho)^2 + N\ddot{\bar{\rho}}/\bar{\rho} + \tfrac{1}{2}N(N-1)(\dot{\bar{\rho}}/\bar{\rho})^2 + 1/\rho^2\\
&\qquad + \tfrac{1}{2}N(N-1)/\bar{\rho}^2 + 2N(\dot{\rho}/\rho)(\dot{\bar{\rho}}/\bar{\rho}) - \tfrac{1}{2}\Lambda = -\tfrac{8}{3}\pi\bar{G}T_{ij}g^{ij}\,,\\
&3\ddot{\rho}/\rho + 3(\dot{\rho}/\rho)^2 + (N-1)\ddot{\bar{\rho}}/\bar{\rho} + \tfrac{1}{2}(N-1)(N-2)(\dot{\bar{\rho}}/\bar{\rho})^2 + 3(N-1)(\dot{\bar{\rho}}/\bar{\rho})(\dot{\rho}/\rho)\\
&\qquad + \tfrac{1}{2}(N-1)(N-2)/\bar{\rho}^2 - \tfrac{1}{2}\Lambda\\
&\qquad + 3/\rho^2 = -\frac{8\pi}{N}\bar{G}T_{\alpha\beta}g^{\alpha\beta}\,.
\end{aligned}
\tag{2.4}
$$

Now, let us calculate the contribution to the energy-momentum tensor. By introducing matter fields or by expanding the metric around the classical ansatz, we can calculate quantum loop corrections which contribute a tower of resonances to the energy-momentum tensor. We will find, however, that the qualitative features of these quantum corrections to the classical ansatz can be roughly duplicated by a single scalar field.

Let us assume that a minimally coupled massive scalar field is interacting with the gravitational field. It is a simple matter to functionally integrate out over this scalar field, which is now defined over the product manifold:

$$
\int \mathcal{D}\varphi\ \mathrm{e}^{iS(\varphi)} = \det{}^{-1/2}\,[-\partial_t^2 + \nabla^2 + m^2(t)]\,. \tag{2.5}
$$

(The factor $m(t)^2$ is a function of the bare mass as well as certain time derivatives of the radii.)

We should note that (2.5) and our basic conclusions are based on the approximation that the "fissioning" process has already terminated and the radius of the three-sphere is already much larger than the radius of the N-sphere, i.e. when several

Planck times have already lapsed. This is because of two reasons. First, quantum effects will dominate the early era when the two radii are of equal magnitude. Thus, our entire mathematical framework breaks down in this early period near $t=0$. Second, we wish to avoid the problem of time-dependent adiabatic effects. At later inflationary times, when the three-dimensional radius is much larger than the N-dimensional radius, time-dependent effects are explicitly included in $m(t)$ in (2.5). We have calculated such effects, and find that they can be neglected if the radius of the three-dimensional universe is much larger than that of the N-dimensional universe. But this is precisely the physical domain which we are studying. In other words, the time-dependent terms in $m(t)$ in (2.5) have been calculated and can be shown to be small in the region we are studying, when the radius of the three-sphere and the logarithmic time derivative of the N-sphere are both larger than the radius of the N-sphere. Thus, we can drop such terms in this paper. (See the last paper in ref. [9].)

We now exponentiate this determinant in order to calculate its contribution to the energy-momentum tensor.

The determinant over the product manifold can be computed as the product over the various eigenvalues of the laplacian. On an N sphere, the ith eigenvalue of the laplacian is given by $i(i+N-1)$ with a degeneracy of $(2i+N-1)\times(i+N-2)!/(N-1)!i!$.

In order to put in the temperature dependence into our formalism, we assume that the scalar field is defined over a periodic time interval with a period given by β:

$$\varphi(x,t)=\sum_k e^{2\pi ikt/\beta}\varphi(x). \tag{2.6}$$

We are making the assumption that the scalar field is a periodic function defined over a finite time interval because it is in thermal equilibrium in a background Robertson–Walker metric. We assume that the background metric, however, is not in equilibrium, i.e. it expands in a normal background de Sitter phase without periodicity. (Introducing temperature into our formalism via periodic functions defined in a non-periodic background gravitational field is also used in the usual GUT formalism.)

Crucial to this assumption is the condition that quantum gravitational corrections, which are of the order of the Planck length, can be neglected. We can implement this assumption if there are, say, several hundred or thousand matter fields coupled to gravity. The dynamics of fissioning and inflation then take place in an energy region several orders of magnitude larger than the Planck length, so we can consistently neglect quantum gravitational effects. Our approximation, however, breaks down in the high-temperature domain, where we can no longer neglect quantum gravitational effects. Therefore, we have focused our discussion on low-temperature effects rather than high-temperature effects, where our approximation may actually break down.

Putting everything together, we find that we can express the energy-momentum in terms of a function V given by:

$$V=\frac{1}{4\pi^2}\sum_{\substack{i,j=0\\k=-\infty}}^{\infty}{}' \,(i+1)^2(2j+N-1)(j+N-2)!/(N-1)!j!$$
$$\times \ln\left\{\frac{i(i+2)}{\rho^2}+\frac{j(j+N-1)}{\bar{\rho}^2}+\frac{4\pi k^2}{\beta^2}\right\}. \tag{2.7}$$

(The prime denotes the deletion of the $i=j=k=0$ term in the summation.)

We find that the components of the energy-momentum tensor can be written as derivatives on the function V:

$$\begin{aligned} T_{00}g^{00}&=-(\rho^3\bar{\rho}^N\beta)^{-1}\beta\,\partial_\beta V,\\ T_{ij}g^{ij}&=-(\rho^3\bar{\rho}^N\beta)^{-1}\rho\,\partial_\rho V,\\ T_{\alpha\beta}g^{\alpha\beta}&=-(\rho^3\bar{\rho}^N\beta)^{-1}\bar{\rho}\,\partial_{\bar{\rho}} V, \end{aligned} \tag{2.8}$$

3. Zeta function regularization

The function V defined earlier formally diverges when we take the sum. In order to regularize the function V, we will find that the zeta functional method is the most convenient of several regularization schemes.

Let us define the following function:

$$\xi(s)=\sum_{\substack{i,j=0\\k=-\infty}}^{\infty}{}' \,(i+1)^2(2j+N-1)(j+N-2)!/(N-1)!j!$$
$$\times\left\{\frac{i(i+2)}{\rho^2}+\frac{j(j+N-1)}{\bar{\rho}^2}+\frac{4\pi k^2}{\beta^2}\right\}^{-s}. \tag{3.1}$$

In terms of this function, we can re-express our function V, and hence all components of the energy-momentum tensor:

$$V=-\frac{1}{4\pi^2}\left(\xi'(0)+\xi(0)\ln\mu^2\right), \tag{3.2}$$

where μ is an arbitrary mass scale.

At this point, the function still diverges in the region of interest. Our next step is to re-express the function in terms of the generalized Epstein zeta function [10], which has a known analytic continuation. Let $(g_1, g_2\cdots g_n)$, $(h_1, h_2\cdots h_n)$ and $(m_1, m_2\cdots m_n)$ represent sequences of n numbers. Let c_{ij} be a symmetric $n\times n$ matrix with an inverse.

Let us define the nth rank Epstein zeta function as:

$$Z\begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ h_1 & h_2 & \cdots & h_n \end{pmatrix}_{\varphi}(s) \equiv \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} e^{2\pi i \sum_{l=1}^{n} m_l h_l} \varphi^{-s/2}((m+g)),$$

$$\varphi((x)) \equiv \sum_{i,j=1}^{n} c_{ij} x_i x_j . \tag{3.3}$$

It is now straightforward to re-express the function ξ totally in terms of derivatives of Epstein zeta functions of third, second, and first rank: (for convenience, we set $N=3$, we rescale $\beta \to 2\pi\beta$ and we let the prime indicate that the lowest term in the summation for m, n, k is deleted; if k goes from minus infinity to positive infinity, then the prime indicates that $k=0$ is deleted)

$$\xi(s) = \sum_{\substack{m,n=1 \\ k=-\infty}}^{\infty}{}' m^2 n^2 \left(\frac{m^2}{\rho^2} + \frac{n^2}{\bar{\rho}^2} + \frac{k^2}{\beta^2} - b^2 \right)^{-s}$$

$$= \sum_{\substack{m,n=1 \\ k=-\infty}}^{\infty}{}' \left\{ (f_{mnk} - b^2)^{-s} - (1-s)^{-1} \left(\rho^4 \frac{\partial}{\partial \rho^2} + \bar{\rho}^4 \frac{\partial}{\partial \bar{\rho}^2} \right) (f_{mnk} - b^2)^{-s+1} \right.$$

$$\left. + \frac{\rho^4 \bar{\rho}^4}{(s-2)(s-1)} \frac{\partial}{\partial \rho^2} \frac{\partial}{\partial \bar{\rho}^2} (f_{mnk} - b^2)^{-s+2} \right\} = \sum_{r=0}^{\infty} A_r + B_r + C_r , \tag{3.4}$$

where

$$f_{mnk} \equiv \frac{m^2}{\rho^2} + \frac{n^2}{\bar{\rho}^2} + \frac{k^2}{\beta^2} \equiv \varphi_3 , \qquad f_{nk} \equiv \frac{n^2}{\bar{\rho}^2} + \frac{k^2}{\beta^2} \equiv \varphi_{\bar{\rho}\beta} ,$$

$$f_{mk} \equiv \frac{m^2}{\rho^2} + \frac{k^2}{\beta^2} \equiv \varphi_{\rho\beta} , \qquad b^2 \equiv \frac{1}{\rho^2} + \frac{1}{\bar{\rho}^2} - m^2(t) , \tag{3.5}$$

$$A_r \equiv \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(1+r)} b^{2r} \sum_{\substack{m,n=1 \\ k=-\infty}}^{\infty}{}' f_{mnk}^{-s-r} = \frac{1}{4} \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(1+r)} b^{2r} \left\{ Z\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\varphi_3} (2s+2r) \right.$$

$$\left. - Z\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{\varphi_{\rho\beta}} (2s+2r) - Z\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{\varphi_{\bar{\rho}\beta}} (2s+2r) + 2\beta^{2s+2r} \zeta(2s+2r) - 4(b^2)^{-s-r} \right\} , \tag{3.6}$$

$$B_r \equiv \tfrac{1}{4}(s-1)^{-1} \left(\rho^4 \frac{\partial}{\partial \rho^2} + \bar{\rho}^4 \frac{\partial}{\partial \bar{\rho}^2} \right) b^{2r} \frac{\Gamma(s+r-1)}{\Gamma(s-1)\Gamma(r+1)} \left\{ Z\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\varphi_3} (2s+2r-2) \right.$$

$$- \left[Z\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{\varphi_{\rho\beta}} (2s+2r-2) + (\rho \leftrightarrow \bar{\rho}) \right] + 2\beta^{2s+2r-2} \zeta(2s+2r-2)$$

$$\left. - 4(b^2)^{-s-r+1} \right\} , \tag{3.7}$$

$$C_r \equiv \frac{1}{4}\frac{\rho^4\bar{\rho}^4}{(s-2)(s-1)}\frac{\partial}{\partial\rho^2}\frac{\partial}{\partial\bar{\rho}^2}\frac{\Gamma(s+r-2)}{\Gamma(s-2)r(1+r)}b^{2r}\left\{Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi_3}(2s+2r-4)\right.$$
$$\left.-\left[Z\begin{pmatrix}0&0\\0&0\end{pmatrix}_{\varphi_{\rho\beta}}(2s+2r-4)+(\rho\leftrightarrow\bar{\rho})\right]+2\beta^{2s+2r-4}\zeta(2s+2r-4)-4(b^2)^{-s-r+2}\right\}. \tag{3.8}$$

Notice that when $s=0$, the function ξ diverges. In order to regularize this function, we now make use of the following relations, which allows us to analytically continue an nth rank zeta function defined at the point s to a point $n-s$:

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)Z\begin{pmatrix}g_1&g_2&\cdots&g_n\\h_1&h_2&\cdots&h_n\end{pmatrix}_{\varphi}(s)$$
$$=\Delta^{-1/2}\exp\left[-2\pi i\sum_{l=1}^{n}g_lh_l\right]\pi^{-(n-s)/2}\Gamma(\tfrac{1}{2}(n-s))$$
$$\times Z\begin{pmatrix}h_1&h_2&\cdots&h_n\\-g_1&-g_2&\cdots&-g_n\end{pmatrix}_{\Phi}(n-s), \tag{3.9}$$

$$\Delta\equiv\det(c_{ij}),$$
$$\Phi\equiv\sum_{i,j=1}^{n}c'_{ij}x_ix_j,$$
$$c'_{ij}\equiv\Delta^{-1}\frac{\partial\Delta}{\partial c_{ij}}. \tag{3.10}$$

At this point, we will still need one more formula, which is the value of the zeta function at $s=3$ and $s=2$. In the appendix, we will prove that the zeta function of fixed rank at a given point can always be re-written in terms of the sums of zeta functions of lower rank. Because zeta functions of first rank are the usual riemannian zeta functions, this means that an Epstein zeta function of arbitrary rank can eventually be reduced to sums over riemannian zeta functions. In the appendix, we will show that:

$$\lim_{s\to2}Z\begin{pmatrix}0&0\\0&0\end{pmatrix}_{\varphi_2}(s)=\frac{2\pi}{\Delta^{1/2}(s-2)}+\frac{\pi^2}{3a}+\frac{2\pi\gamma}{\Delta^{1/2}}+\pi\Delta^{-1/2}\ln\frac{a}{4\Delta}$$
$$-\frac{2\pi}{\Delta^{1/2}}\sum_{n=1}^{\infty}\ln(1-q_1^{2n})(1-q_2^{2n}), \tag{3.11}$$

$$\lim_{s\to3}Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi_3}(s)$$
$$=\frac{4\pi}{\rho\bar{\rho}\beta(s-3)}+\frac{2\zeta(3)}{\bar{\rho}^3}+\frac{2\pi^2}{3\rho^2\bar{\rho}}+\frac{4\pi(\gamma-1)}{\rho\bar{\rho}\beta}-\frac{4\pi\ln\beta}{\rho\bar{\rho}\beta}-\frac{8\pi}{\rho\bar{\rho}\beta}\sum_{n=1}^{\infty}\ln(1-e^{-2\pi\beta n/\rho})$$

$$-\frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty} n^{-3}\left\{\frac{1}{\rho^3}\ln\left(1-e^{-4\pi\rho n/\bar{\rho}}\right)+\frac{1}{\beta^2}\ln\left(1-e^{-4\pi\beta n/\bar{\rho}}\right)\right\}$$

$$+\frac{4\sqrt{2}}{\bar{\rho}}\sum_{n=1}^{\infty} n^{-2}\left\{\frac{1}{\rho^2}\left[e^{4\pi\rho n/\bar{\rho}}-1\right]^{-1}+\frac{1}{\beta^2}\left[e^{4\pi\beta n/\bar{\rho}}-1\right]^{-1}\right\}$$

$$+\frac{8\sqrt{2}}{\bar{\rho}}\sum_{m,k=1}^{\infty}\left(\rho^2 m^2+\beta^2 k^2\right)^{-1}\left(e^{B_{mk}}-1\right)^{-1}$$

$$-\frac{2\sqrt{2}}{\pi}\sum_{m,k=1}^{\infty}\left(\rho^2 m^2+\beta^2 k^2\right)^{-3/2}\ln\left(1-e^{-B_{mk}}\right), \tag{3.12}$$

$$\varphi_2 \equiv a m_1^2 + 2b m_1 m_2 + c m_2^2, \qquad \Delta = ac - b^2,$$

$$q_1 \equiv \exp \pi i\left(-\frac{b}{a}+i\frac{\sqrt{\Delta}}{a}\right), \qquad q_2 \equiv \exp \pi i\left(-\frac{b}{a}-i\frac{\sqrt{\Delta}}{a}\right),$$

$$B_{mk} \equiv \frac{4\pi}{\bar{\rho}}\left(\rho^2 m^2+\beta^2 k^2\right), \qquad \gamma \equiv 0.57721566\ldots. \tag{3.13}$$

We now have all the tools necessary to re-express the function V entirely in terms of simple summations and known analytic functions. The final calculation is straightforward but quite tedious. We will present the final result in the appendix.

We will find, however, that in the regions of interest, almost all the terms drop out and the potential function reduces to a remarkably simple result.

3. Topological inflation

At this point, we have reduced the function V to a series of known functions and easily computable summations.

Although the problem of computing V is now formally solved, we will find it convenient to make certain approximations which will allow us to compute the cosmological behavior of our equations.

We assume self-consistency in region II, that the radius of the 3-dimensional universe is much larger than the N-dimensional universe. We also assume that the mass terms are small so we can drop the mass term $m(t)$ in our potential (we will return to this point later). In this approximation, we can reduce the expression for the zeta function to the following form:

$$\lim_{\rho\to\infty} Z\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_{\varphi}(s)$$

$$= 2\rho^s\zeta(s) + 2\sqrt{\pi}\,\Gamma((s-1)/s)\,\Gamma^{-1}(\tfrac{1}{2}s)\,\rho\beta^{s-1}\zeta(s-1) + \frac{\rho\beta\,\Gamma((s-2)/s)}{\pi^2\Gamma(\tfrac{1}{2}s)}\,\bar{\rho}^{s-2}\zeta(s-2)$$

$$+\frac{8\rho\beta\pi^{s/2}}{\Gamma(\tfrac{1}{2}s)}\left(\beta\bar{\rho}k/n\right)^{(s-2)/2}K_{(s-2)/2}\left(2\pi\beta k n/\bar{\rho}\right). \tag{4.1}$$

We prove this formula in the appendix. Given this expression, we can now express the entire potential V in terms of simple summations and Bessel functions in the limit of large ρ:

$$-4\pi^2 V = \tfrac{2}{45}\pi^4 \frac{\rho^3}{\beta^3} + \frac{\rho^3\beta}{\bar{\rho}^4}\left\{-\frac{\zeta(3)}{32\pi^2} - \frac{3\zeta(5)}{16\pi^4} - \tfrac{45}{64}\pi^{-6}\zeta(7) + \frac{1}{8}\sum \frac{n!\zeta(2n)}{(n+3)!}\right\} + \left(\frac{\rho}{\bar{\rho}}\right)^3 \sum_{\substack{i=0\\ j,k=1}}^{\infty} (j^{i-2}\alpha^{1-i}k^{4-i}/i!)K_{i-2}\left(\frac{jk\beta}{\rho}\right)\left(\frac{\beta}{\rho}\right)^{i-1}. \tag{4.2}$$

At this point, we can now begin the search for cosmological solutions to the equations of motion.

In region I, where the time scale is many orders of magnitude smaller than the Planck length, we find that the manifold M^{3+N} executes a simple de Sitter expansion.

In region II, we assume that the time scale is sufficiently large enough to activate the spontaneous fissioning of the $3+N$ dimensional manifold into a $S^3 \times S^N$ product manifold. The 3-dimensional universe then executes a de Sitter expansion, while the other universe collapses down to the Planck scale.

In this region, we find that the expression for V simplifies even further to:

$$V = C_3\rho^3\beta/\bar{\rho}^4 - \tfrac{1}{90}\pi^2(\rho/\beta)^3, \qquad C_3 \equiv 7.5688\cdots\times 10^{-5}. \tag{4.3}$$

This compares favorably with the Candelas-Weinberg static result of $C_3 = 7.5687\times 10^{-5}$, which was calculated using an entirely different regularization scheme.

Given this simple expression for V, we can now re-express our original equations of motion into the final form:

$$\begin{aligned}
&3x^2+3/\bar{\rho}^2-\tfrac{1}{2}\Lambda+[5y^2+\dot{y}+9xy+3/\rho^2] = 24\pi G\bar{\rho}_0^3(\tfrac{1}{30}\pi^2(\bar{\rho})^{-3}\beta^{-4}+C_3/\bar{\rho}^7),\\
&2\dot{x}+3x^2+3/\bar{\rho}^2-\tfrac{1}{2}\Lambda+[3\dot{y}+6y^2+1/\rho^2] = 8\pi G\bar{\rho}_0^3(-\tfrac{1}{30}\pi^2(\bar{\rho})^{-3}\beta^{-4}+3C_3/\bar{\rho}^7),\\
&3\dot{x}+6x^2+[3y^2+2\dot{y}+3/\rho^2]-\tfrac{1}{2}\Lambda+1/\bar{\rho}^2 = 8\pi G\bar{\rho}_0^3(-4C_3/\bar{\rho}^7),\\
&x\equiv\dot{\rho}/\rho, \qquad y\equiv\dot{\bar{\rho}}/\bar{\rho}.
\end{aligned} \tag{4.4}$$

In region II, let us assume that $\rho\to\infty$ and $\bar{\rho}=$ constant. Then all factors in the brackets vanish in (4.4).

A solution for these equations exists if we pursue the analogy with the usual GUT picture. In many ways, these equations mimic the GUT inflation equations generated by Higgs particles. In some sense, the equations of motion for the M^N radius resemble the equations of motion for the Higgs particle, i.e. they are simple newtonian equations driven by a potential function. (In our picture, however, the potential contains derivative terms which are equivalent, in the newtonian picture, to friction-like terms.)

Using the newtonian-GUT picture as a guide, we can find the following solution for the equations of motion in region II, where the M^3 radius undergoes a de Sitter expansion while the M^N radius is fixed:

$$\begin{aligned}
&\bar{\rho} = (1.46033\cdots)\bar{\rho}_0\,,\\
&\rho \sim \exp\left[0.515305\cdots(t/\bar{\rho}_0)\right],\\
&y = \dot{y} = 0\,, \qquad \beta^{-1} = 0\,,\\
&\bar{G} = G\bar{\rho}_0^3\,, \qquad \bar{\rho}_0 = (0.047109\cdots)G^{1/2}\,.
\end{aligned} \tag{4.5}$$

Notice that this de Sitter expansion eventually terminates on its own. When the radius of the N-dimensional universe begins to move away from its value of $(1.40633\cdots)\bar{\rho}_0$, it gradually approaches its equilibrium value at $\bar{\rho}_0$. When the radius of the N-dimensional universe finally reaches its equilibrium value, then the de Sitter expansion terminates.

In region III, the three-dimensional universe undergoes a simple big bang driven by the matter content of the energy-momentum tensor. The N-dimensional universe has now settled down to its equilibrium value of $\bar{\rho}_0$ while the three-dimensional universe undergoes a simple power law expansion (big bang)

$$\begin{aligned}
&\rho \sim t^{1/2}\,, \qquad \Lambda = \tfrac{30}{7}\bar{\rho}_0^{-2}\,,\\
&\bar{\rho} = \bar{\rho}_0\,, \qquad \beta \sim t^{1/2}\,.
\end{aligned} \tag{4.6}$$

We have checked explicitly on computer that the inflation phase can be made to extend for arbitrarily long periods of time and that it eventually terminates into a big bang phase.

Notice that the value of the primordial cosmological constant used in region I is now determined by the equilibrium equations of region III. In order for the inflation to terminate smoothly into the big bang phase of region III, it is necessary to fine tune the cosmological constant at the beginning of time. Our picture, therefore, does not give a solution to the cosmological constant problem. Once the cosmological constant is fixed at the beginning of time, however, our scenario explains why the effective cosmological constant at the present time is so small. Because the Planck length is the only dimensionful parameter in the theory and because considerably many Planck lengths have passed since the original fissioning, the effective cosmological constant found today must be astronomically small. Our picture, then, actually makes a prediction as to the present value of the cosmological constant. Unfortunately, the precise value of the effective cosmological constant at the present time is too small to be reliably calculated.

The next question we ask is how many parameters are necessary to achieve the e^{65} factor necessary to explain the horizon and flatness problems. In the usual GUT picture, there are certain difficulties in fine tuning the parameters to achieve the proper e^{65} expansion factor.

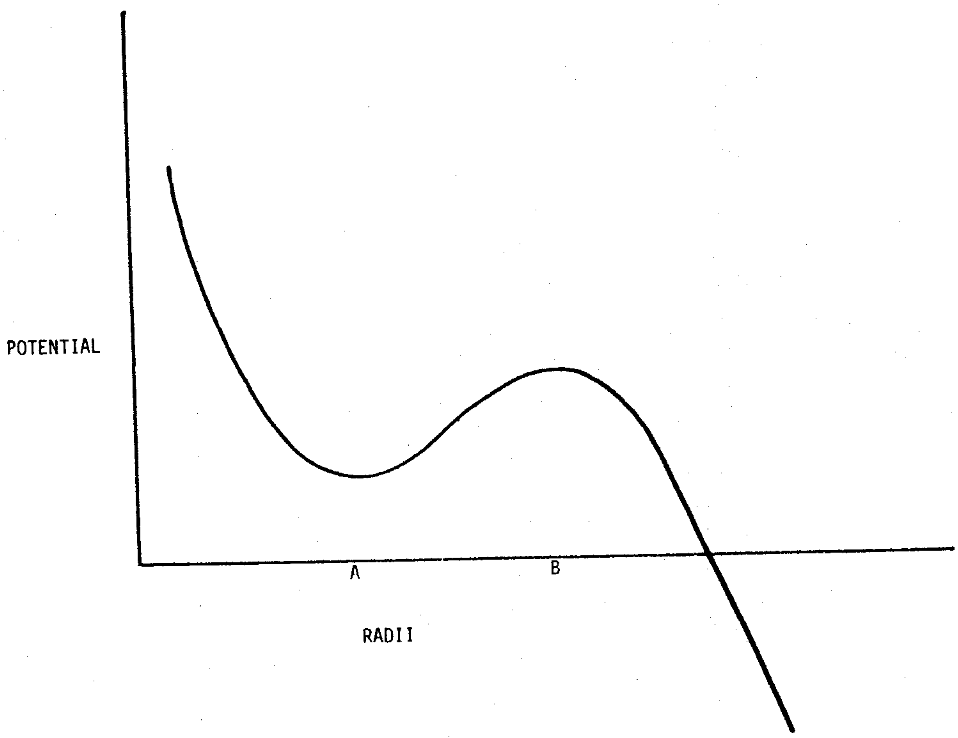

Fig. 2. As in the case with GUT inflation, the evolution of the "topological inflation" phase depends on the time it takes for a ball to roll down a potential curve. Because we have the freedom to choose the values of the initial parameters, such as the mass of the scalar particles, we can flatten this curve and vary the time period over which inflation takes place. (Without a proper choice of initial parameters, however, the ball would have to be placed directly on top of the hill, at point B, to insure that we get enough inflation. This can conceivably re-introduce the fine tuning problem.) The shape of the curve is quite dependent on a variety of factors, such as the mass. Unfortunately, we do not have a satisfactory mechanism which determines how the universe fissions, and therefore do not have the initial parameters which will determine how much inflation takes place.

If we examine the equations for the radius of the N-dimensional universe, we notice that they resemble the equations for a newtonian particle in a potential well, similar to the situation in GUT inflation scenarios with Higgs particles. The inflation takes place when the radius of the N-dimensional universe is placed close to its unstable equilibrium value of $(1.40633\cdots)\bar{\rho}_0$, which lies on a hill-top in potential space. If we take the mass to zero and set all friction-like terms to zero, then we have a simple potential curve as in fig. 2.

Inflation takes place while the particle is on the top of the hill as well as while it rolls down the hill. Inflation terminates when the particle finally asymptotically oscillates around the very bottom of the hill. If there are no mass terms $m(t)$, then we must precisely fine tune the position of the particle directly on top of the hill, which is undesirable.

In reality, of course, there are several parameters to play with. First of all, the mass term for the scalar field changes the shape of the original potential curve by flattening out the hill top and therefore increases the time interval in which inflation takes place.

For small mass, the net effect is to lengthen the time of inflation. The shape of the potential is now a function of a continuous parameter m. For large masses, however, it may actually be possible to significantly alter the shape of the potential curve, including the addition of more than one local minimum and maximum. We will explain this in more detail in another paper.

Second, we have the time dependent terms coming from the original evaluation of the $3+N$ dimensional laplacian. These friction terms, although they vanish in the region of interest (at the top of a local maximum) contribute when the particle rolls down the hill.

Third, we can still adjust the parameters coming in from the original fissioning process, i.e. the initial value of the radii and its derivatives when fissioning took place.

5. Conclusion

In this paper, we are proposing an alternative to both the usual Kaluza-Klein and the standard GUT inflation pictures. When we reverse the Kaluza-Klein scenario, and allow three dimensions to peel off from a $3+N$ dimensional compact manifold, we find, remarkably enough, that the three-dimensional manifold undergoes a de Sitter-like expansion, while the other N-dimensional manifold collapses down to the Planck length. By varying certain parameters, we have enough freedom to achieve an arbitrary amount of inflation, enough to explain the horizon and flatness problems. In a later paper, we will elaborate on precisely how these parameters affect the amount of inflation.

The application of the Kaluza-Klein formalism to cosmology, therefore, leads to a consistent picture compatible with the known features of the early universe.

Our method stands in contrast to other time-dependent formalisms involving Kaluza-Klein theory. Our method uses quantum one-loop corrections to generate the effective cosmological term and to drive the inflation. Alternatively, one might conceivably use external factors, such as higher derivative terms, the equation of state for a classical gas, or even an external monopole [11-13] to generalize the static Kaluza-Klein picture. We find, however, that these terms unnecessarily complicate the model, and not all of them are compatible with inflation. Single-loop corrections, in any event, are an essential feature of any Kaluza-Klein model regardless of external features one may add, and single-loop corrections alone are sufficient to give satisfactory cosmological behavior. Single-loop corrections generate acceptable cosmological behavior by themselves without adding external features.

Again, we stress that we certainly have not solved the fine tuning problem. In some sense, what we are doing is replacing our ignorance of where inflation comes

from with our ignorance of how the original universe was spontaneously or dynamically broken. Before we can realistically calculate exactly how much inflation our scenario produces, we must know the initial radii of the two universes after fissioning. This, in turn, implies that we must wait until we understand symmetry breaking schemes before we can say with confidence precisely how much inflation this model produces. In this sense, we are replacing one set of unknown parameters with another.

Note added

After this paper was written, we found that M. Yoshimura [14] also investigated the time-dependent behavior of Kaluza–Klein theory when single-loop corrections are included. He suggests that the mechanism of oscillating Newton's constant may help to understand the horizon and flatness problems in the high-temperature limit. Our paper analyzes the behavior at relatively low temperatures, where we find inflation. (We avoided an elaborate discussion of the high-temperature limit in our work because the fundamental approximation in our work, that quantum effects from gravity itself can be dropped, begins to break down in the high-temperature limit.)

Appendix A

In this appendix, we analyze the decomposition of the V potential in terms of zeta functions. Using the identities presented earlier in the paper, we can reduce the potential function V to the following form:

$$V=\sum_{r=0}^{\infty}\tilde{A}_r+\tilde{B}_r+\tilde{C}_r\,, \tag{A.1}$$

where

$$\tilde{A}_0\equiv\Omega-\frac{\pi\beta}{12}\left(\frac{1}{\rho}+\frac{1}{\bar{\rho}}\right)+\ln\left(\sqrt{2\pi\beta}\,b^2\right)+\sum_{n=1}^{\infty}\ln\left(1-e^{-2\pi\beta n/\rho}\right)\left(1-e^{-2\pi\beta n/\bar{\rho}}\right), \tag{A.2}$$

$$\begin{aligned}\Omega\equiv{}&\frac{\zeta(3)\rho\beta}{4\pi\bar{\rho}^2}+\frac{\pi\beta}{12\rho}-\tfrac{1}{2}\ln(2\pi\beta)-\sum_{n=1}^{\infty}\left(\ln\left(1-e^{-2\pi\beta n/\rho}\right)\right.\\&-\frac{\sqrt{2}}{8\pi^2}\frac{\bar{\rho}\beta}{\rho^2}\sum_{n=1}^{\infty}n^{-3}\ln\left(1-e^{-4\pi\rho n/\bar{\rho}}\right)-\frac{\sqrt{2}\rho\bar{\rho}}{8\pi^2\beta^2}\sum_{n=1}^{\infty}n^{-3}\ln\left(1-e^{-4\pi\beta n/\bar{\rho}}\right)\\&+\frac{\beta}{\sqrt{2}\pi\rho}\sum_{n=1}^{\infty}n^{-2}\left(e^{4\pi\rho n/\bar{\rho}}-1\right)^{-1}+\frac{\rho}{\sqrt{2}\pi\beta}\sum_{n=1}^{\infty}n^{-2}\left(e^{4\pi\beta n/\bar{\rho}}-1\right)^{-1}\\&+\frac{\sqrt{2}\rho\beta}{\pi}\sum_{m,k=1}^{\infty}b_{mk}^{-1}\left(e^{B_{mk}}-1\right)^{-1}-\frac{\sqrt{2}}{4\pi^2}\sum_{m,k=1}^{\infty}\rho\bar{\rho}\beta b_{mk}^{-3/2}\ln\left(1-e^{-B_{mk}}\right),\end{aligned} \tag{A.3}$$

$$\tilde{A}_1=\tfrac{1}{4}b^2 Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi_3}(2)-\left(\tfrac{1}{4}b^2 Z\begin{pmatrix}0&0\\0&0\end{pmatrix}_{\varphi_{\rho\beta}}(2)+(\rho\leftrightarrow\bar{\rho})\right)+\tfrac{1}{12}\pi^2\beta^2 b^2+\ln b^2\,, \tag{A.4}$$

$$\tilde{A}_r(r>1)=\frac{1}{r}\,b^{2r}\sum_{\substack{m,n=1\\k=-\infty}}^{\infty}{}' f^{-r}_{mnk}+\left(\tfrac{1}{4}\sum_{m,k=-\infty}^{+\infty}{}' f^{-r}_{mk}+(\rho\leftrightarrow\bar{\rho})\right)$$
$$-\sum_{k=-\infty}^{\infty}{}'\left(\frac{k^2}{\beta^2}\right)^{-r}+(b^2)^{-r}\,, \tag{A.5}$$

$$\tilde{B}_0=\frac{3\rho\bar{\rho}\beta(\rho^2+\bar{\rho}^2)}{32\pi^3}\sum_{m,n,k=-\infty}^{\infty}{}' f^{-5/2}_{mnk}-\frac{15\rho\bar{\rho}\beta}{32\pi^3}\sum_{m,n,k=-\infty}^{\infty}(\rho^4 m^2+\bar{\rho}^4 n^2)f^{-7/2}_{mnk}$$
$$+\left\{\frac{2\rho\beta}{4\pi^3}\sum_{m,k=1}^{\infty}\rho^4 m^2 f^{-3}_{mk}+\frac{\zeta(4)\beta}{\rho\pi^3}-\frac{\beta\rho^3}{2\pi^3}\sum_{m,k=1}^{\infty}f^{-2}_{mk}-\frac{\beta\rho^3}{4\pi^3}\left(\frac{1}{\rho^4}+\frac{1}{\beta^4}\right)\zeta(4)\right.$$
$$\left.+(\rho\leftrightarrow\bar{\rho})\right\}+2\ln b^2\,, \tag{A.6}$$

$$\tilde{B}_1=\left\{-2+b^2\left(\rho^4\frac{\partial}{\partial\rho^2}+\bar{\rho}^4\frac{\partial}{\partial\bar{\rho}^2}\right)\right\}\Omega+\left\{-\ln(2\pi\beta)+\frac{\pi\beta}{b\rho}+\frac{\pi\beta\rho b^2}{24}\right.$$
$$\left.-2\sum_{n=1}^{\infty}\ln(1-e^{-2\pi\beta n/\rho})-\pi\beta\rho b^2\sum_{n=1}^{\infty}n(e^{2\pi\beta n/\rho}-1)^{-1}+(\rho\leftrightarrow\bar{\rho})\right\}$$
$$-\zeta(0)\ln\beta^2-2\zeta'(0)-2\ln b^2-2\,, \tag{A.7}$$

$$\tilde{B}_2=\frac{1}{8}\left(\rho^4\frac{\partial}{\partial\rho^2}+\bar{\rho}^4\frac{\partial}{\partial\bar{\rho}^2}\right)b^4 Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi_3}-\tfrac{1}{6}\pi^2\beta^2 b^2+1\,, \tag{A.8}$$

$$\tilde{B}_r(r>2)=\frac{1}{4r(r-1)}\left(\rho^4\frac{\partial}{\partial\rho^2}+\bar{\rho}^4\frac{2}{\partial\bar{\rho}^2}\right)b^{2r}Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi_3}(2r-2)\,, \tag{A.9}$$

$$\tilde{C}_0=\frac{15}{32\pi^5}\left\{\tfrac{63}{4}\rho^5\bar{\rho}^5\beta\sum_{m,n,k=-\infty}^{\infty}{}' m^2 n^2 f^{-11/2}_{mnk}-\tfrac{7}{4}\rho^3\bar{\rho}^3\beta\sum_{m,n,k=-\infty}^{\infty}{}'(\rho^2 m^2+\bar{\rho}^2 n^2)f^{-9/2}_{mnk}\right.$$
$$\left.+\tfrac{1}{4}\rho^3\bar{\rho}^3\beta\sum_{m,n,k=-\infty}^{\infty}f^{-7/2}_{mnk}\right\}+\ln b^2\,, \tag{A.10}$$

$$\tilde{C}_1=\frac{3}{16\pi^3}\left\{(\tfrac{1}{4}\rho^3\bar{\rho}^3\beta b^2-\tfrac{1}{2}\rho\bar{\rho}\beta(\rho^2+\bar{\rho}^2))\sum_{m,n,k=-\infty}^{\infty}{}' f^{-5/2}_{mnk}\right.$$
$$-\tfrac{5}{2}(\tfrac{1}{2}\rho\bar{\rho}^3\beta b^2-\rho\bar{\rho}\beta)\sum_{m,n,k=-\infty}^{\infty}{}'\rho^4 m^2 f^{-7/2}_{mnk}-\tfrac{5}{2}(\tfrac{1}{2}\bar{\rho}\rho^3\beta b^2-\rho\bar{\rho}\beta)\sum_{m,n,k=-\infty}^{\infty}{}'\bar{\rho}^4 n^2 f^{-7/2}_{mnk}$$
$$\left.+\tfrac{35}{4}\rho^5\bar{\rho}^5\beta b^2\sum_{m,n,k=-\infty}^{\infty}{}' m^2 n^2 f^{-9/2}_{mnk}\right\}$$

$$+\left\{\frac{\rho^3\beta}{2\pi^3}\sum_{m,k=1}^{\infty} f_{mk}^{-2}-\frac{2\rho^5\beta}{\pi^3}\sum_{m,k=1}^{\infty} m^2 f_{mk}^{-3}\right.$$

$$\left.+\frac{1}{4\pi^3}\zeta(4)\rho^3\beta(\bar\rho^4+\beta^{-4})-\frac{\zeta(4)\beta}{\rho\pi^3}+(\rho\leftrightarrow\bar\rho)\right\}-2\ln b^2-1\,, \tag{A.11}$$

$$\tilde C_2=\left(1-b^2\left(\rho^4\frac{\partial}{\partial\rho^2}+\bar\rho^4\frac{\partial}{\partial\bar\rho^2}\right)\right)\Omega+\tfrac14 b^2\left\{\frac{\sqrt2}{16\pi^2}\bar\rho^3\beta\sum_{n=1}^{\infty} n^{-3}\ln\left(1-e^{-4\pi\rho n/\bar\rho}\right)\right.$$

$$-\frac{\sqrt2}{8\pi}\rho\bar\rho^2\beta\sum_{n=1}^{\infty} n^{-2}\left(e^{4\pi\beta n/\bar\rho}-1\right)^{-1}-\frac{\sqrt2}{4\pi}\rho\bar\rho^2\beta\sum_{n=1}^{\infty} n^{-2}\left(e^{4\pi\rho n/\bar\rho}-1\right)^{-1}$$

$$-\frac{\sqrt2\rho\bar\rho^4\beta}{32\pi^2}\sum_{n=1}^{\infty} n^{-3}\left[\frac{16\pi^2 n^2}{\bar\rho^3}\frac{e^{4\pi\rho n/\bar\rho}}{(e^{4\pi\rho n/\bar\rho}-1)^2}-\frac{4\pi n/\bar\rho^2}{e^{4\pi\rho n/\bar\rho}-1}\right]$$

$$-\frac{\rho^2\bar\rho\beta}{\sqrt2}\sum_{n=1}^{\infty} n^{-1}\frac{e^{4\pi\rho n/\bar\rho}}{(e^{4\pi\rho n/\bar\rho}-1)^2}+\frac{\sqrt2}{8\pi}\rho^2\bar\rho^3\beta\sum_{n=1}^{\infty} n^{-2}\left[\frac{4\pi n}{\bar\rho^2}\frac{e^{4\pi\rho n/\bar\rho}}{(e^{4\pi\rho n/\bar\rho}-1)^2}\right.$$

$$\left.+\frac{(16\pi^2\rho n^2/\bar\rho^2)\,e^{4\pi\rho n/\bar\rho}}{(e^{4\pi\rho n/\bar\rho}-1)^2}-\frac{32\pi^2\rho n\,e^{8\pi\rho n/\bar\rho}}{\bar\rho^3(e^{4\pi\rho n/\bar\rho}-1)^3}\right]-\sqrt2\rho^5\bar\rho\beta\sum_{m,k=1}^{\infty}\frac{m^2\,e^{B_{mk}}}{f_{mk}^{3/2}(e^{B_{mk}}-1)^2}$$

$$+\frac{\sqrt2\rho^4\bar\rho^3\beta}{4\pi}\sum_{m,k=1}^{\infty} f_{mk}^{-1}\left[\frac{16\pi^2\rho m^2\,e^{B_{mk}}}{\bar\rho^4(e^{B_{mk}}-1)^2}-\frac{32\rho m^2\,e^{2B_{mk}}}{\bar\rho^3(e^{B_{mk}}-1)^3}\right]$$

$$-\frac{\sqrt2\rho^3\bar\rho^3\beta}{16\pi^2}\sum_{m,k=1}^{\infty} f_{mk}^{-3/2}\ln\left(1-e^{-B_{mk}}\right)+\frac{\sqrt2\rho^3\bar\rho^2\beta}{4\pi}\sum_{m,k=1}^{\infty} f_{mk}^{-1}\left(e^{B_{mk}}-1\right)^{-1}$$

$$-\frac{\sqrt2}{16\pi^2}\rho^4\bar\rho^3\beta\sum_{m,k=1}^{\infty}\left[f_{mk}^{-5/2}(-3\rho m^2)\ln\left(1-e^{-B_{mk}}\right)+f_{mk}^{-2}\frac{4\pi\rho m^2}{\bar\rho(e^{B_{mk}}-1)}\right]$$

$$\left.-\frac{\sqrt2}{16\pi^2}\rho^4\bar\rho^4\beta\sum_{m,k=1}^{\infty}\left\{\frac{12\pi\rho m^2}{\bar\rho^2 f_{mk}^2(e^{B_{mk}}-1)}+f_{mk}^{-3/2}\frac{16\pi^2\rho m^2\,e^{B_{mk}}}{\bar\rho^3(e^{B_{mk}}-1)^2}-\frac{4\pi\rho m^2}{\bar\rho^2 f_{mk}^2(e^{B_{mk}}-1)}\right\}\right\}$$

$$+\tfrac12\ln(2\pi\beta)-\frac{\pi\beta}{12\rho}+\sum_{n=1}^{\infty}\ln\left(1-e^{-2\pi\beta n/\rho}\right)$$

$$-b^2\left[-\rho^3\pi\beta/\bar\rho^2\sum_{n=1}^{\infty} n\left(e^{2\pi\beta n/\bar\rho}-1\right)^{-1}+\tfrac{1}{24}\pi\beta\rho+(\rho\leftrightarrow\bar\rho)\right]$$

$$-\frac{\sqrt2}{32\pi^2}\frac{\rho^3\bar\rho^3}{\beta^2}\sum_{n=1}^{\infty} n^{-3}\ln\left(1-e^{-4\pi\beta n/\bar\rho}\right)+\frac{\sqrt2}{8\pi}\frac{\rho^3\bar\rho^2}{\beta}\sum_{n=1}^{\infty} n^{-2}\left(e^{4\pi\beta n/\bar\rho}-1\right)^{-1}$$

$$-\tfrac14\ln\beta^2-\tfrac12\ln 2\pi+\ln\beta^2+\tfrac32\,, \tag{A.12}$$

$$\tilde C_3=\tfrac{1}{24}\rho^4\bar\rho^4\frac{\partial}{\partial\rho^2}\frac{\partial}{\partial\bar\rho^2}b^6 Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi_3}(2)+\tfrac{1}{12}\pi^2\beta^2b^2-\tfrac13\,. \tag{A.13}$$

Appendix B

In this appendix, we will show how to express an Epstein zeta function of third rank in terms of zeta functions of lower rank. This method can be generalized to zeta functions of arbitrary rank.

The starting point is the following identity, which we shall use throughout our calculation:

$$\sum_{m=-\infty}^{\infty} e^{-\pi z m^2} = \frac{1}{\sqrt{z}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/z} . \tag{B.1}$$

We shall also use the following expression for the Epstein zeta function:

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s) Z\begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ h_1 & h_2 & \cdots & h_n \end{pmatrix}_{\varphi}(s)$$
$$= \int_0^{\infty} dz\, z^{s/2-1} \sum_{m_1} \cdots \sum_{m_n} \exp\left[-\pi z \varphi((g+m)) + 2\pi i \sum_{l=1}^{n} m_l h_l\right] . \tag{B.2}$$

Let us begin our discussion by defining the following summation:

$$f(m_1 m_2 m_3 z) \equiv \sum_{m_1, m_2, m_3} \exp\left[-\pi z(a m_1^2 + b m_2^2 + c m_3^2)\right] - 1 ,$$

$$Z\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\varphi}(s) = \pi^{s/2}\Gamma^{-1}(\tfrac{1}{2}s) \int_0^{\infty} dz\, z^{s/2-1} f(m_1 m_2 m_3 z) . \tag{B.3}$$

We now make repeated use of the first identity to re-express the above function in the following form:

$$\begin{aligned} f(m_1 m_2 m_3 z) = {} & \frac{1}{\sqrt{cz}} \sum_{m_1, m_2, m_3}{}' \exp\left[-\pi a z m_1^2 - \pi b z m_2^2 - \pi m_3^2/cz\right] \\ & + \frac{1}{\sqrt{cz}} \sum_{m_2, m_3}{}' \exp\left[-\pi b z m_2^2 - \pi m_3^2/cz\right] \\ & - \frac{1}{\sqrt{cz}} \sum_{m_1, m_3}{}' \exp\left[-\pi a z m_1^2 - \pi m_3^2/cz\right] \\ & + \frac{1}{\sqrt{cz}} \frac{1}{\sqrt{az}} \sum_{m_1, m_2}{}' \exp\left[-\pi m_1^2/az - \pi z b m_2^2\right] \\ & + \sum_{m_3}{}' e^{-\pi z c m_3^2} + \frac{1}{\sqrt{cz}} \sum_{m_1}{}' e^{-\pi z a m_1^2} \\ & + \sum_{m_2}{}' e^{-\pi z b m_2^2} (\sqrt{cz}\sqrt{az})^{-1} . \end{aligned} \tag{B.4}$$

We now multiply the previous expression by $z^{(s/2-1)}$ and integrate:

$$\begin{aligned}
&\pi^{-s/2}\Gamma(\tfrac{1}{2s})Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi}(s)\\
&\quad=2\pi^{-s/2}c^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s)\\
&\qquad+\frac{2}{\sqrt{c}}\,\pi^{-(s-1)/2}a^{-(s-1)/2}\Gamma(\tfrac{1}{2}(s-1))\zeta(s-1)\\
&\qquad+\frac{2}{\sqrt{ac}}(\pi b)^{-(s-2)/2}\Gamma(\tfrac{1}{2}(s-2))\zeta(s-2)\\
&\qquad+\frac{4}{\sqrt{ac}}\int_0^\infty \mathrm{d}z\, z^{((s-2)/2-1)}\sum_{m_1,m_2=1}^{\infty}\exp\left[-\pi m_1^2/az-\pi zbm_2^2\right]\\
&\qquad+\frac{4}{\sqrt{c}}\int_0^\infty \mathrm{d}z\, z^{((s-1)/2-1)}\sum_{m_1,m_3=1}^{\infty}\exp\left[-\pi azm_1^2-\pi m_3^2/cz\right]\\
&\qquad+\frac{4}{\sqrt{c}}\int_0^\infty \mathrm{d}z\, z^{((s-1)/2-1)}\sum_{m_2,m_3=1}^{\infty}\exp\left[-\pi bzm_2^2-\pi m_3^2/cz\right]\\
&\qquad+\frac{8}{\sqrt{c}}\int_0^\infty \mathrm{d}z\, z^{((s-1)/2-1)}\sum_{m_1,m_2,m_3=1}^{\infty}\exp\left[-\pi zam_1^2-\pi zbm_2^2-\pi m_3^2/cz\right].
\end{aligned}\tag{B.5}$$

All integrals can be re-expressed simply in the following form:

$$\int_0^\infty \mathrm{d}z\, z^{\alpha-1}\,\mathrm{e}^{-z-t/z}=2^{2\alpha+1}\sqrt{\pi}\,t^{\alpha}\,\mathrm{e}^{-2\sqrt{t}}\,\Gamma^{-1}(\alpha+\tfrac{1}{2})\int_0^\infty \mathrm{d}z\,(z+z^2)^{\alpha-1/2}\,\mathrm{e}^{-4z\sqrt{t}}.\tag{B.6}$$

The right-hand side of the previous expression can be easily integrated for various values of α. (When $\alpha=1$, the integral yields a combination of Bessel functions $I_{3/2}-I_{-3/2}$.)

All integrations can now be explicitly performed. When terms are collected, we arrive at (3.12).

Appendix C

In this part, we will calculate the limit of the zeta function when one of the radii goes to infinity.

We start with the definition of the zeta function:

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi}(s)=\int_0^\infty \mathrm{d}z\, z^{s/2-1}\left[\sum_{m,n,k=-\infty}^{\infty}\exp\left[-\pi z\left(\frac{m^2}{\rho^2}+\frac{n^2}{\bar{\rho}^2}+\frac{k^2}{\beta^2}\right)\right]-1\right].\tag{C.1}$$

Using the identity which flips exponentials of z into $1/z$, we find:

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)Z\begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}_{\varphi}(s)$$

$$=2\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\rho^{s}\zeta(s)+2\pi^{-((s-1)/2)}\Gamma(\tfrac{1}{2}(s-1))\rho\beta^{s-1}\zeta(s)$$

$$+2\pi^{-((s-2)/2)}\Gamma(\tfrac{1}{2}(s-2))\rho\beta\bar{\rho}^{s-2}\zeta(s-2)$$

$$+\frac{2\rho\beta}{\pi}\int_0^\infty \mathrm{d}z\, z^{((s-2)/2-1)}\sum_{n,k=-\infty}^{\infty}\exp\left[-\pi z n^2/\rho^2-\pi^2\beta^2k^2/z\right]$$

$$+4\rho\int_0^\infty \mathrm{d}z\, z^{((s-2)/2-1)}\sum\exp\left[-\pi z h^2/\rho-\pi\rho^2 m^2/z\right]$$

$$+\frac{2\rho\beta}{\pi}\int_0^\infty \mathrm{d}z\, z^{((s-2)/2-1)}\sum_{m,n=-\infty}^{\infty}\exp\left[-\pi z n^2/\bar{\rho}^2-\pi\rho^2m^2/z\right]$$

$$+\frac{4\rho\beta}{\pi}\int_0^\infty \mathrm{d}z\, z^{((s-2)/2-1)}\sum_{m,n,k=-\infty}^{\infty}\exp\left[-\pi z n^2/\bar{\rho}^2-\pi\rho^2m^2/z-\pi\beta^2k^2/z\right]. \tag{C.2}$$

Now we use the formula that the modified Bessel function can be written as:

$$\int_0^\infty \mathrm{d}z\, z^{((s-2)/2-1)}\exp\left[-\pi z n^2/\rho^2-\pi\beta^2k^2/z\right]$$

$$=2(\beta\bar{\rho}k/n)^{(s-2)/2}K_{((s-2)/2)}(2\pi\beta kn/\bar{\rho}). \tag{C.3}$$

Putting everything together, we find our final result (4.1).

References

[1] Th. Kaluza, Sitz. Preuss. Akad. Wiss. K1 (1921) 966;
O. Klein, Z. Phys. 37 (1926) 895
[2] Z. Horvath, L. Palla, E. Cremmer and J. Scherk, Nucl. Phys. 127 (1977) 57
[3] M. Kaku and J. Lykken, The fissioning universe: a Kaluza-Klein solution to the horizon and flatness problem, CCNY-HEP-21-83 (Dec. 1983)
[4] A. Guth, Phys. Rev. D23 (1981) 347;
A.D. Linde, Phys. Lett. 108B (1982) 389;
A. Albrecht and P. Steinhardt, Phys. Rev. Lett. 48 (1982) 1220
[5] P.J. Steinhardt, The current state of the inflationary universe, Univ. of Penn. preprint
[6] P.G.O. Freund and M.A. Rubin, Phys. Lett. 97B (1980) 233
[7] G.W. Gibbons and H. Nicolai, CERN preprint TH-3815 (Feb. 1984)
[8] P. Candelas and D.J. Raine, Nucl. Phys. B243 (1984) 157
[9] P. Candelas and S. Weinberg, Nucl. Phys. B237 (1984) 397;
M.A. Rubin and B.D. Roth, Nucl. Phys. B226 (1983) 444;
G. Gilbert, B. McClain and M.A. Rubin, Phys. Lett. B142 (1984) 28
[10] P. Epstein, Math. Ann. 56 (1903) 615
[11] Q. Shafi and C. Wetterich, Phys. Lett. 129B (1983) 387
[12] D. Shadev, Univ. of Penn. preprint UPR-0238T (1983)
[13] Y. Okada, Univ. of Tokyo preprint UT-429 (March 1984)
[14] M. Yoshimura, KEK-TH 75